CHAPTER 13

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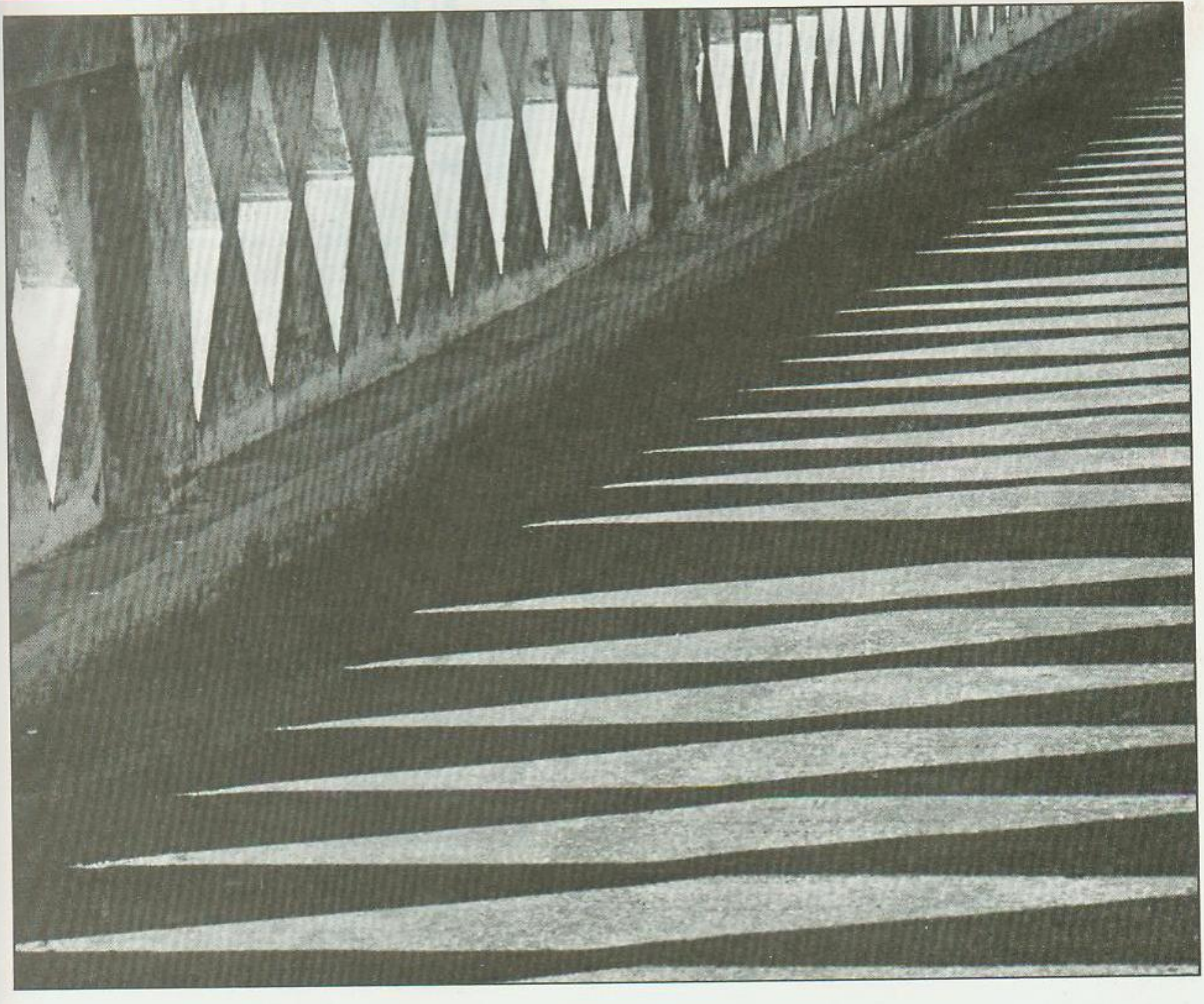
Transformations and Symmetry



13–1 Line Reflections

The word transformation implies that an object is being changed in some way. In a geometric transformation there are three things to consider:

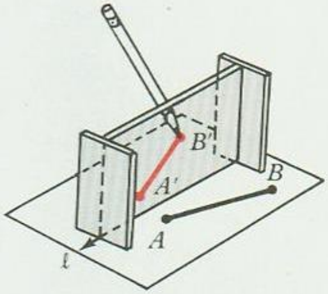
1. the original figure,
2. a rule or operation that describes the change, and
3. the figure that results after that change.

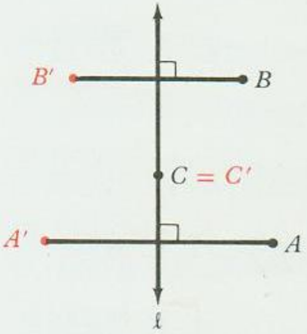
The object prior to the change is called the *preimage,* and the object after the change is called the *image.*

In this chapter we shall study the three types of transformations called *line reflections, translations,* and *rotations.* The first of these transformations, a line reflection, can be described by using a MIRA.

Drawing by W. Miller;
© 1962 The New Yorker Magazine, Inc.

Suppose a MIRA is placed on a line ℓ as shown. Points A' and B' are the reflection images of A and B.





Notice that ℓ is the perpendicular bisector of $\overline{AA'}$ and $\overline{BB'}$. This is true for any segment connecting a point and its reflection image. Point C, because it is on ℓ, is its own image.

Definition 13-1

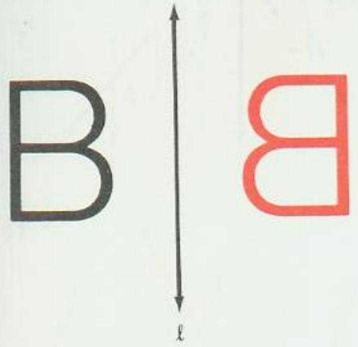
In a plane, a **reflection** over line ℓ is a transformation that maps each point P of the plane onto the point P' as follows:

a. If P is on ℓ, $P' = P$.

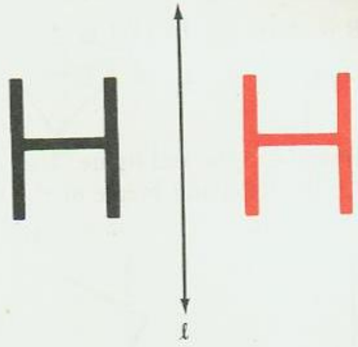
b. If P is not on ℓ, then ℓ is the perpendicular bisector of $\overline{PP'}$.

P' is called the **image** of P and P is the **preimage** of P'.

When each point of a figure is reflected over a line ℓ, the set of all image points form a figure called the reflection image of the figure. Two examples are shown.



A letter "B" and its reflection over line ℓ.



A letter "H" and its reflection over line ℓ.

The transformation called a line reflection satisfies several important properties as stated in this theorem.

Theorem 13–1 Given a line reflection:

a. the reflection image of a segment is a segment of equal length;

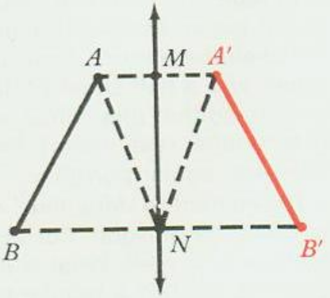
b. the reflection image of an angle is an angle of equal measure.

Below is the plan for proving part a of Theorem 13–1.

PROOF

Given: $\overline{AB}$ is the reflection image of $\overline{A'B'}$.
Prove: $AB = A'B'$

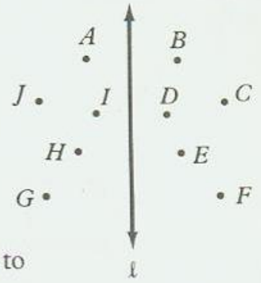
Plan: ℓ is the perpendicular bisector of $\overline{AA'}$ and $\overline{BB'}$. Draw in auxiliary segments AN and $A'N$ and use congruent triangles.



The proof is left as an exercise.

EXERCISES

A.



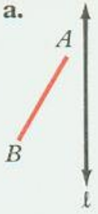
1. The reflection image over line ℓ

   a. of B is $\underline{?}$.   b. of I is $\underline{?}$.

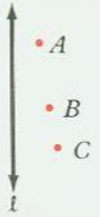
   c. of D is $\underline{?}$.   d. of F is $\underline{?}$.

2. Trace line ℓ and the red figure. Use a compass or a MIRA to construct the reflection image of the red figure.

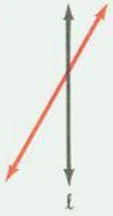
a.



b.

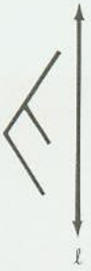


c.

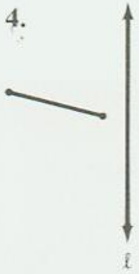


Copy the following figures. Use a MIRA, tracing paper, or a compass to draw as accurately as possible the reflection image of each figure over line ℓ.

3.



4.



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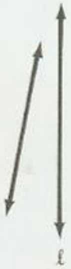
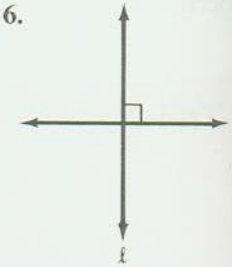
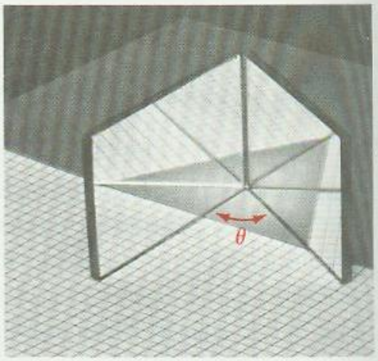


6.



Activity



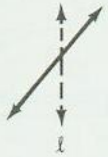
Obtain a pair of rectangular mirrors and hinge them together with tape. You can see different regular polygons by using graph paper and construction paper and changing the angle θ between the mirrors. First, tape together a half sheet of graph paper and a half sheet of dark construction paper. Position the mirrors so that one mirror is at a right angle to a line on the graph paper (see photo). Swing the second mirror to form different regular polygons.

Experiment to determine what values of θ will create a picture of a regular polygon. For example, in the photo an equilateral triangle is formed. What is the value of θ? What value of θ will create a square? a regular pentagon? a regular hexagon? Can you find a relationship between the angle θ and the number of sides?

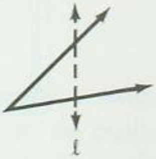
B.

For exercises 7–14, trace the given figure. Make a freehand sketch of the reflection image over line ℓ of the given figure. Check your work with a compass, MIRA, or mirror.

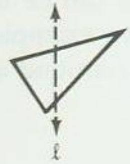
7.



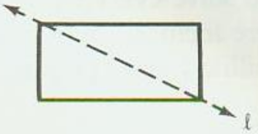
8.



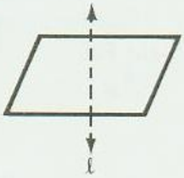
9.



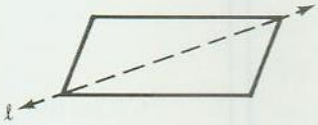
10.



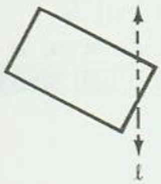
11.



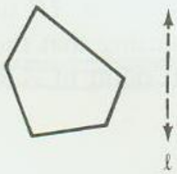
12.



13.

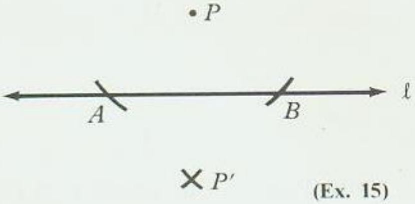


14.



C.

15. A fixed compass construction of the reflection image of a point is shown. As a result of this method, $AP = PB = AP' = BP'$. How do you know that ℓ is the perpendicular bisector of $\overline{PP'}$?



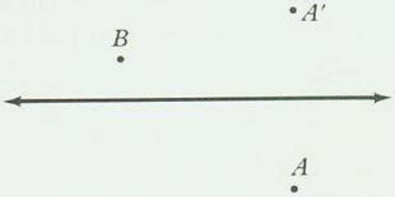
(Ex. 15)

16. Trace this figure in which A' is the reflection image of A. Using only a straightedge, construct the reflection of point B.

17. Write a complete two-column proof that shows that your construction in exercise 16 is correct.

18. Write a two-column proof of Theorem 13–1.

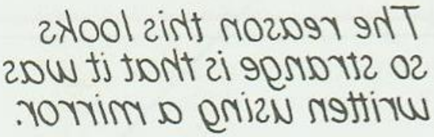
19. Prove that a triangle and its reflection image are congruent.



(Ex. 16)

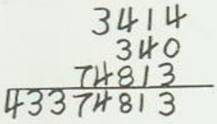
PROBLEM SOLVING

1. Can you decode this message?

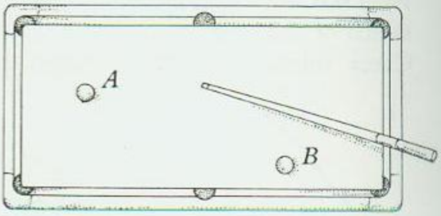


The reason this looks
so strange is that it was
written using a mirror.

2. Is this sum correct?


3414
340
74813
43374813

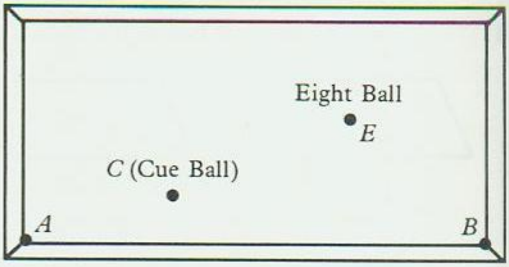
13–2 Using Line Reflections in Problem Solving



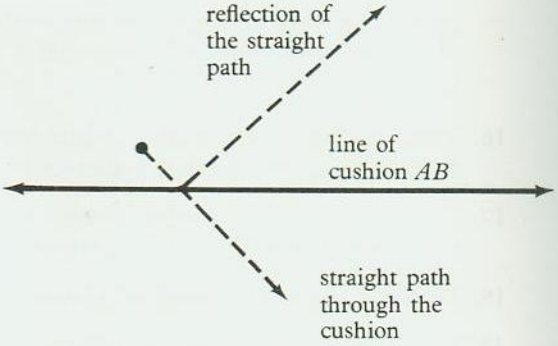
The geometry of line reflections can be used to solve everyday problems. Two examples are included here, one centering around a billiard table problem.

Problem 1

On a billiard table a cue ball is to be banked off cushion $\overline{AB}$ to hit the eight ball E. Assuming that the cue ball has no spin, at what point of $\overline{AB}$ should the cue ball strike?



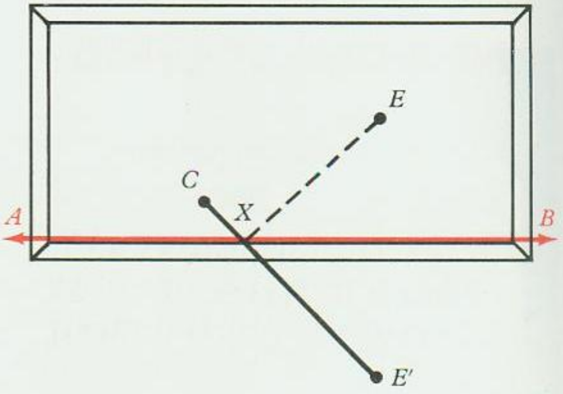
Solution. The answer to this question uses the following fact. *The ball rebounds along a path that is the reflection* (in the line of the cushion) *of the straight path through the cushion.*


reflection of the straight path
line of cushion AB
straight path through the cushion

Step 1 Reflect E over the line AB to point E'.

Step 2 Draw the line CE'. Let X be the point where $\overleftrightarrow{CE'}$ intersects the cushion.

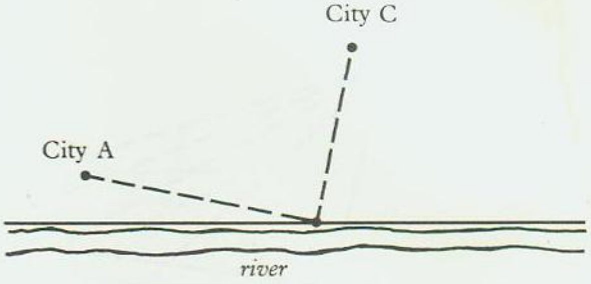
Step 3 A ball hit with no spin at the point X will rebound and hit the eight ball E. Why?



In the first problem we used a line reflection to determine the location of a point that satisfied a necessary condition. The second problem also involves determining the location of a point. In Problem 2 we will use the definition of betweenness for points together with a line reflection to determine a location for a bridge that is to be built.

Problem 2

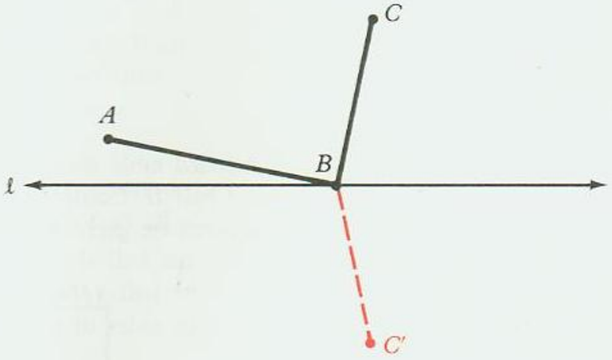
Two cities are positioned on the same side of a river across which a bridge is to be built. Where should the bridge (B) be built so that the length of the highway $AB + BC$ is as short as possible?



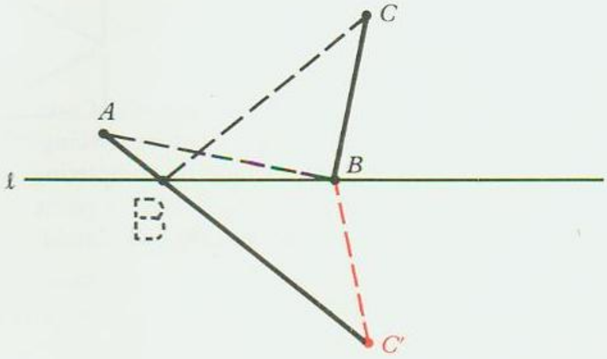
Solution

Step 1 Imagine that the river is line ℓ and reflect point C over line ℓ to point C'.

Step 2 From Theorem 13–1 we can conclude that $BC = BC'$. Therefore $AB + BC = AB + BC'$.



Step 3 But the shortest path from A to C' is a straight path. Therefore, in order for path A-B-C to be a minimum length, the bridge B should be built so that A, B, and C' are collinear.

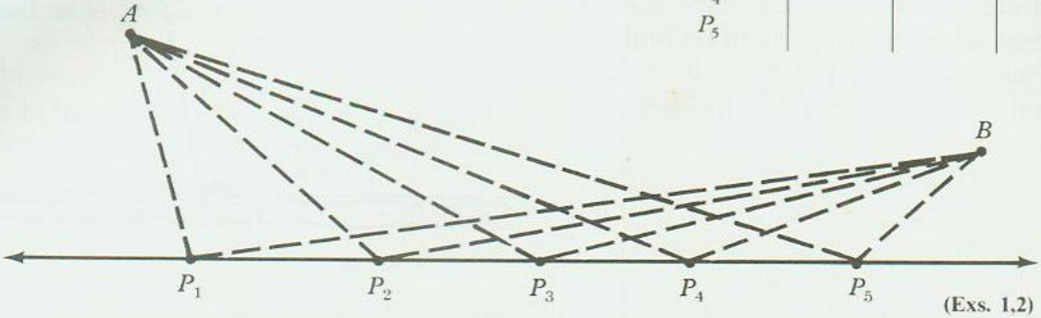


EXERCISES

A.

1. Use a millimeter ruler to measure each of the distances in this figure and record the distances in a table as shown on the right.

Point P_i	AP_i	BP_i	$AP_i + BP_i$
P_1			
P_2			
P_3			
P_4			
P_5			

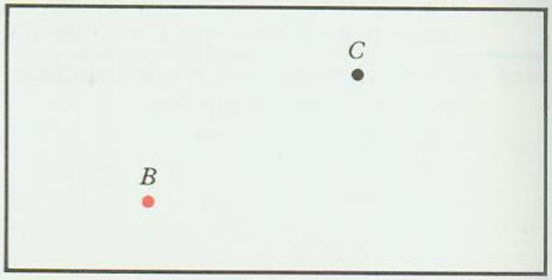


(Exs. 1,2)

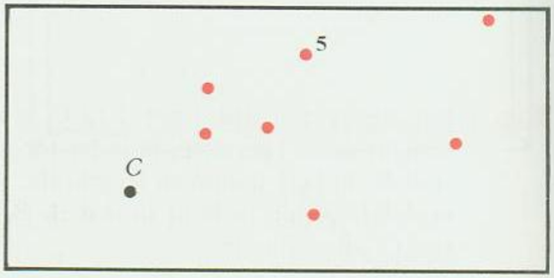
2. In the figure which point, P_1, P_2, P_3, P_4, or P_5, makes $AP_i + P_iB$ a minimum?

B.

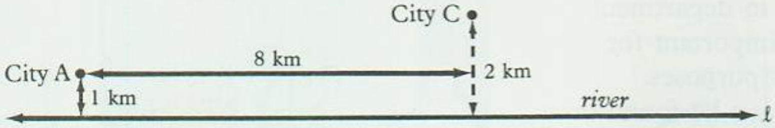
3. Trace the figure of a billiard table with cue ball C and a second ball B. Complete constructions to find a point on each of the four cushions that the cue ball should strike to rebound and hit the ball. (*Hint:* Reflect B in each of the four sides of the table.)



4. The cue ball C is to rebound off of one cushion and strike ball 5 before striking any other ball. Determine by completing a construction the cushion and the point on that cushion that the cue ball should strike.



5. In this figure two cities are shown on the same side of the river. Locate the position (B) where a bridge should be built in order to minimize the length of a highway $AB + BC$.

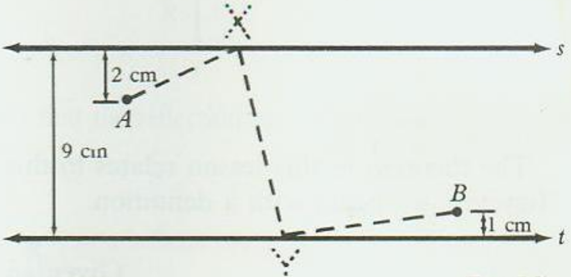


(Exs. 5,6)

6. After constructing the point B in exercise 5, use the Pythagorean Theorem to calculate the minimum length $AB + BC$.

C.

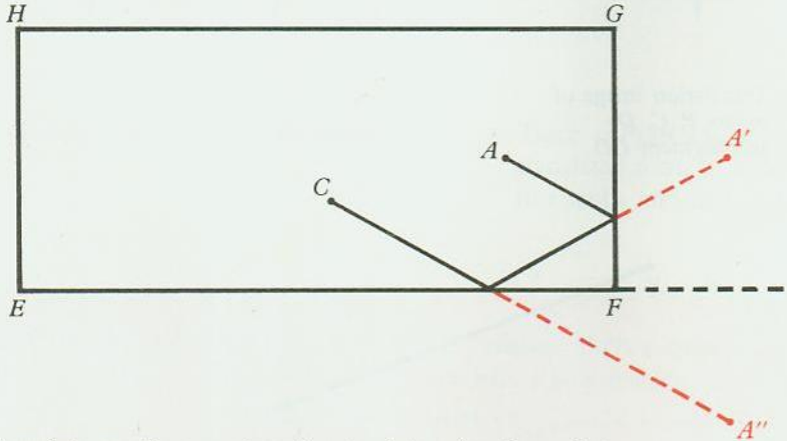
7. Suppose a path from A to B must first touch line s and then line t. Trace this figure and construct points X on s and Y on t so that $AXYB$ is the shortest of all such paths. (*Hint:* Reflect point A over line s and point B over line t.)



(Exs. 7,8)

8. Suppose a path from A to B must first touch line t and then line s. Trace the figure and construct points X on t and Y on s so that $AYXB$ is the shortest of all such paths. How does the length of this path compare to the length of the one found in exercise 7?

9. The figure illustrates how to determine the point on cushion EF that the cue ball C should strike in order to rebound off of *two* cushions and strike ball A.

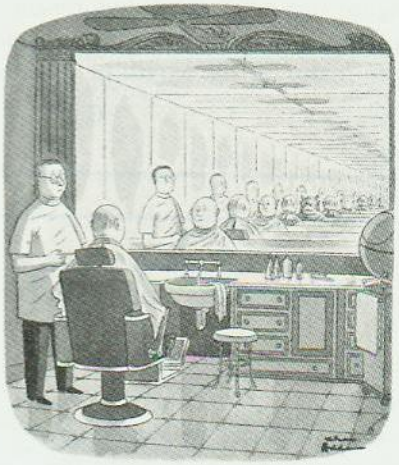


Complete another construction to determine the point on cushion GH that the cue ball should strike in order to rebound off two cushions and strike ball A.

13–3 Translations

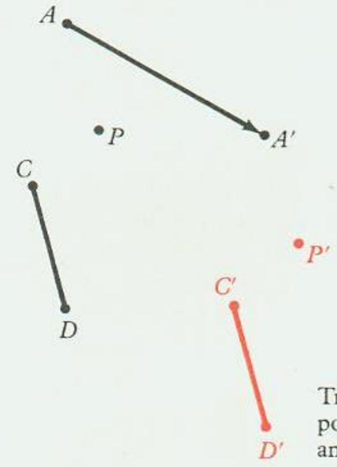
Mirrors can often be found in department stores and shops. They are important for both decorative and security purposes.

Sometimes these mirrors can be found on opposite walls of a room. They have certainly produced an interesting effect in the barbershop in this cartoon.



Drawing by Chas. Addams;
© 1957 The New Yorker Magazine, Inc.

The theorem in this lesson relates to this reflection of a reflection. However, we begin with a definition.



Translation image of points P, C, D and segment CD

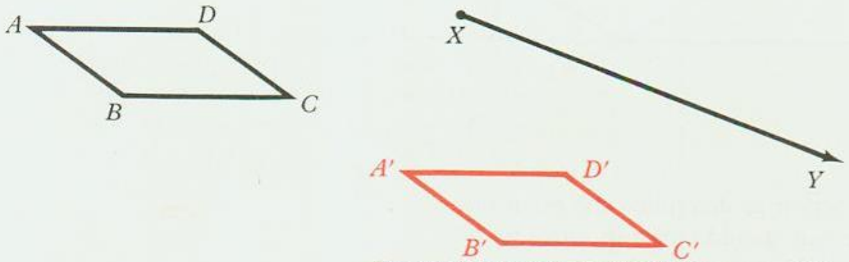
Given an arrow AA', we imagine sliding a figure in the direction of the arrow through a distance AA'.

Definition 13-2

Given an arrow AA', the **translation image** of a point P for the arrow AA' is the point P' where:

a. $AA' = PP'$, and

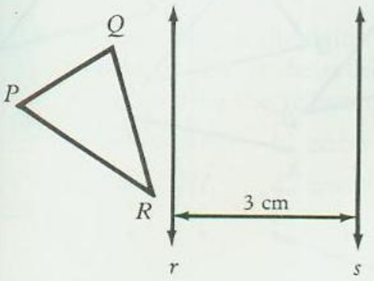
b. arrows AA' and PP' have the same direction.



Translation image of $ABCD$ for arrow XY

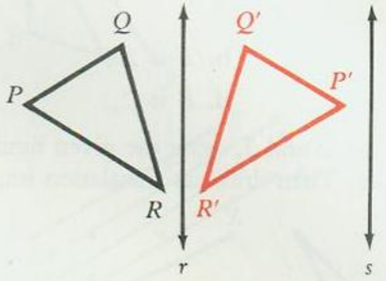
We can accomplish a translation by following one reflection by another. Complete this four-step construction for yourself.

Step 1



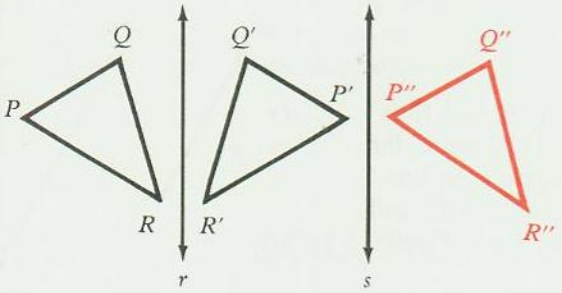
Start with lines r and s parallel and 3 cm apart.

Step 2



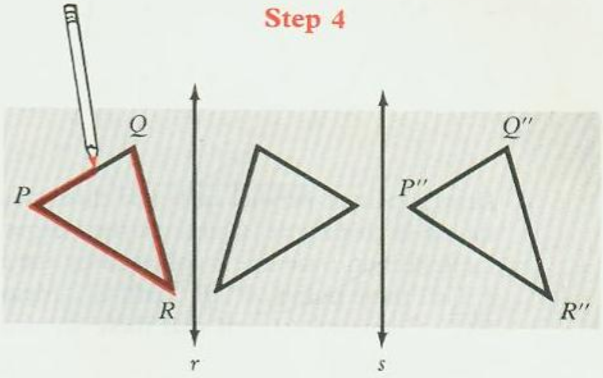
First find the reflection of $\triangle PQR$ over line r.

Step 3



Then find the reflection of $\triangle P'Q'R'$ over line s.

Step 4



Trace $\triangle PQR$ and show that it can be translated 6 cm in a direction perpendicular to r and s and that $\triangle PQR \cong \triangle P''Q''R''$.

Theorem 13–2 If lines r and s are parallel, then a reflection over line r followed by a reflection over line s is a translation. Furthermore, if A'' is the image of A, then

a. $\overline{AA''} \perp r$,

b. $AA'' = 2d$ where d is the distance between the lines r and s.

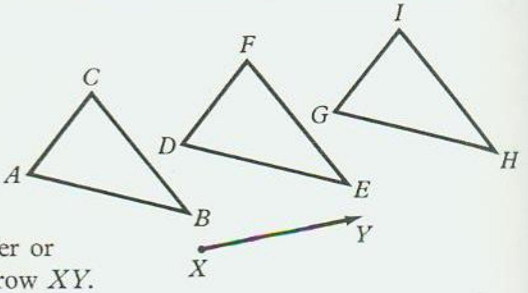
EXERCISES

A.

1. For the translation arrow XY the translation image of:

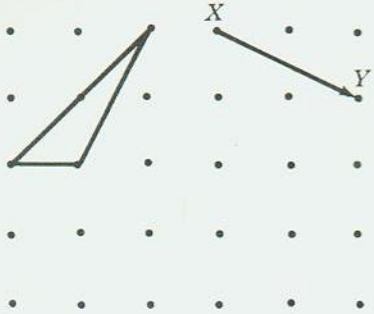
   a. C is _?_.   **b.** E is _?_.

   c. D is _?_.   **d.** B is _?_.

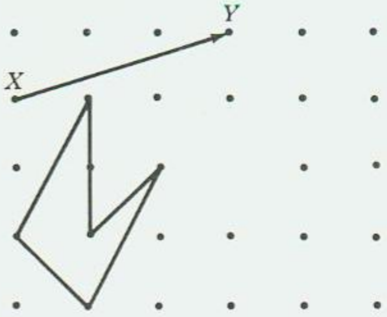


For exercises 2 and 3, draw the given figure on dot paper or graph paper. Then draw its translation image for the arrow XY.

2.

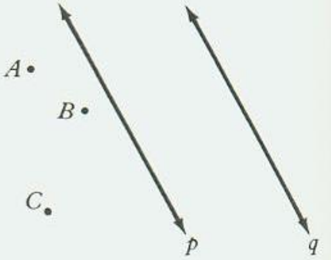


3.



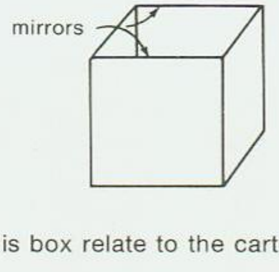
B.

4. Draw a pair of parallel lines and three points A, B, and C as shown in this figure. Construct the images of A, B, and C for the reflection over line p followed by the reflection over line q. Call these points A'', B'', and C''. Measure $\overline{AA''}$, $\overline{BB''}$, and $\overline{CC''}$. Convince yourself that these segments are parallel and equal in length.

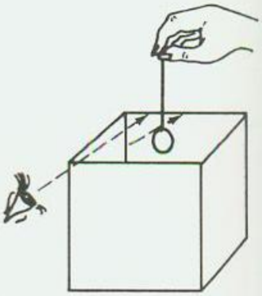


Activity

Cover the front and back inside faces of a box, preferably nearly a cube in shape, with mirrors. Cover the other faces with black construction paper.

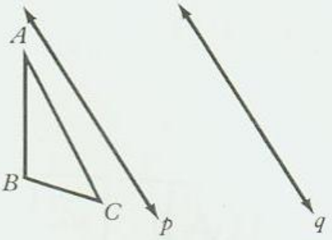


Dangle an object by a string into the box and look into the box over the front edge. What will you see?



How does the view in this box relate to the cartoon on page 484?

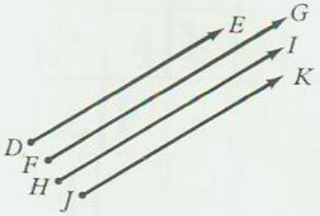
5. Draw a pair of parallel lines p and q that are 3 cm apart and $\triangle ABC$ as shown. Construct the image of $\triangle ABC$ for the reflection over line p followed by the reflection over line q.



6. Which arrow in the figure describes the translation that is the reflection over line p followed by the reflection over line q?

   a. arrow DE **b.** arrow FG

   c. arrow HI **d.** arrow JK



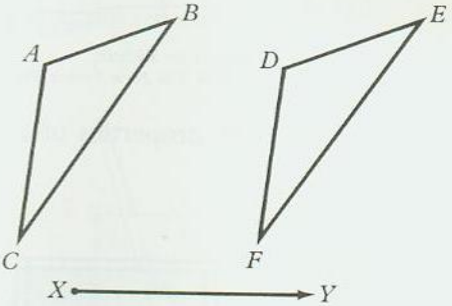
C.

7. In this figure $\triangle DEF$ is the translation image of $\triangle ABC$ for the translation of arrow XY.

   a. Trace the figure on a sheet of paper.

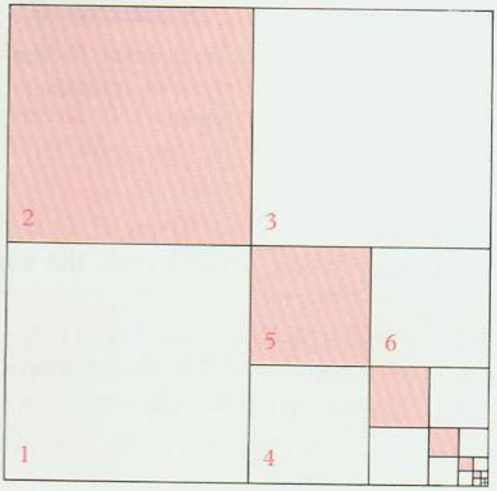
   b. Draw any line p perpendicular to line XY.

   c. Draw a line q so that the reflection over line p followed by the reflection over line q is the translation with arrow XY.

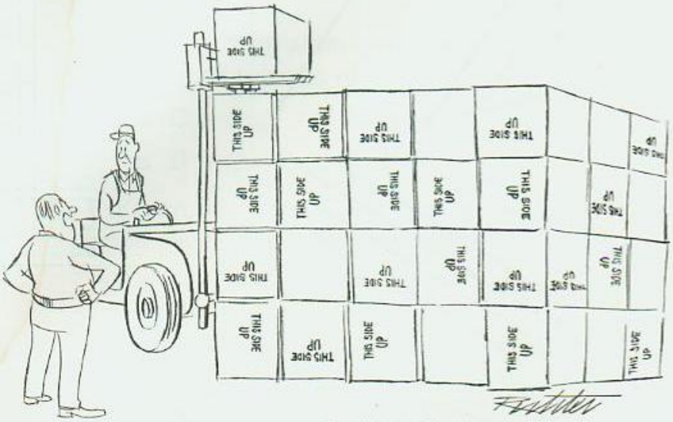


PROBLEM SOLVING

What fraction of the square region is shaded? (Assume that the indicated process of shading is continued indefinitely.)



13-4 Rotations



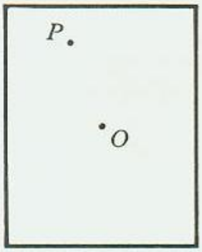
"*You're fired!*"

Drawing by Richter;
© 1957 The New Yorker Magazine, Inc.

There are many situations in which we encounter a "turning situation," often called a "rotation." Failure to recognize the need for this motion may have had serious consequences for the gentleman in the cartoon!

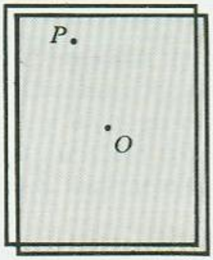
You can investigate the properties of a rotation as shown below.

Step 1



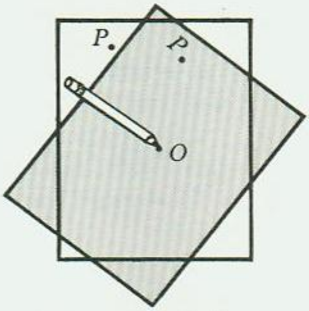
Mark a center point O and another point P on a sheet of paper.

Step 2



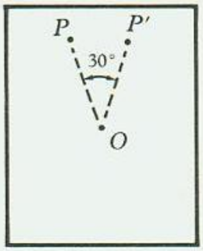
Trace point P on a sheet of tracing paper.

Step 3



Keep point O from moving and turn the tracing paper.

Step 4



Mark the new position of P as P'.

P' is the rotation image of point P. O is the center of the rotation. In this case, the angle of rotation is 30°.

These ideas suggest the accompanying definition of a rotation. Note that a rotation can be clockwise or counterclockwise.

Definition 13-3

A **rotation** with center O and angle α is a transformation that maps each point P of the plane onto a point P' as follows:

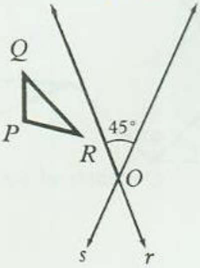
a. If P is the center point O, $P' = P$.

b. If $P \neq O$, then $P'O = PO$ and $m\angle POP' = \alpha$.

P' is called the **rotation image** of point P.

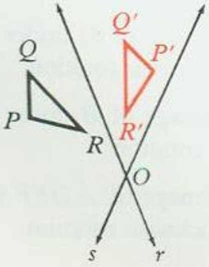
Like translations, rotations can be accomplished by following one reflection by another. This time the lines of reflection are not parallel. Complete this four-step construction for yourself.

Step 1



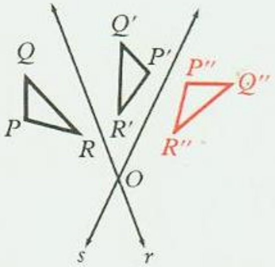
Start with lines r and s, passing through O and intersecting at a 45° angle, and $\triangle PQR$.

Step 2



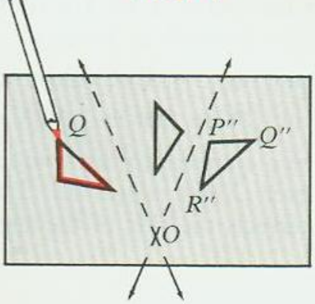
First find the reflection of $\triangle PQR$ over line r.

Step 3



Then find the reflection of $\triangle P'Q'R'$ over line s.

Step 4



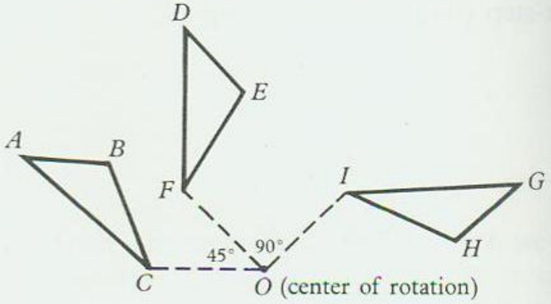
Trace $\triangle PQR$ and show that it can be rotated through a 90° angle around point O and that $\triangle PQR \cong \triangle P''Q''R''$.

Two reflections over intersecting lines will always result in a rotation through an angle twice the angle between the lines. This suggests the theorem below.

Theorem 13-3 If lines r and s intersect in point O, then a reflection over r followed by a reflection over s is a rotation. Point O is the center of the rotation and the angle of rotation is 2α, where α is the measure of the acute or right angle between lines r and s.

EXERCISES

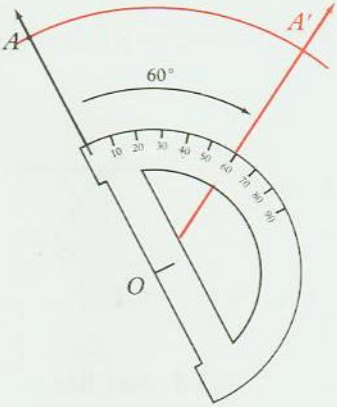
A.



(Exs. 1-4)

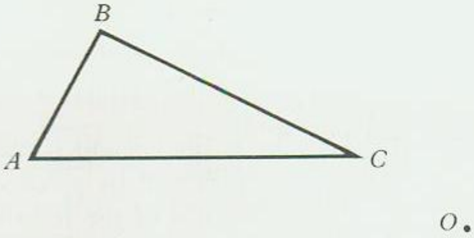
1. Find the image of B under a 45° clockwise rotation.
2. Find the image of H under a 90° counterclockwise rotation.
3. Find the image of B under a 135° clockwise rotation.
4. Find the image of $\triangle DEF$ under a 45° counterclockwise rotation.

5. Draw on your paper points O and A. Draw the image A' of A for a 60° clockwise rotation as follows.
   - **a.** Draw ray $\overrightarrow{OA}$. Use a protractor to draw $\overrightarrow{OX}$ so that $m\angle AOX = 60$.
   - **b.** Use a compass to draw an arc with center O and radius OA. This arc intersects the side of the angle at point A'.



6. Draw a segment $\overline{XY}$ and a point O on your paper. Use a protractor and compass to draw the image of $\overline{XY}$ for a 40° counterclockwise rotation.

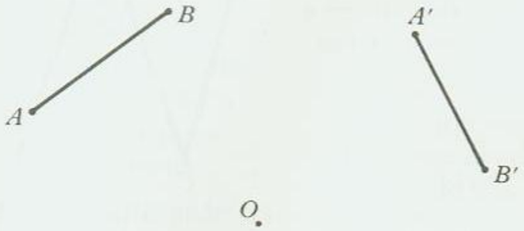
7. Draw a triangle ABC and a point O on your paper. Use a protractor and compass to draw the image of $\triangle ABC$ for a 50° clockwise rotation.



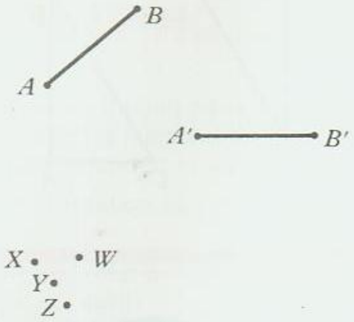
8. Draw a triangle ABC and a point O. Draw the image of $\triangle ABC$ for a 135° counterclockwise rotation.

B.

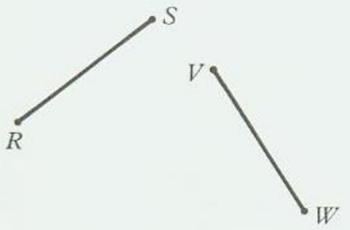
9. If $\overline{A'B'}$ is the image of $\overline{AB}$ for a rotation with center O, which angle should be measured to find the angle of rotation?
   - a. $\angle AOB$
   - b. $\angle AA'O$
   - c. $\angle A'B'O$
   - d. $\angle BOB'$



10. If $\overline{A'B'}$ is the image of $\overline{AB}$ after a rotation, which one of the four points could be the center of rotation? (Recall that if O is the center, $OA = OA'$ and $OB = OB'$.)
    - a. W
    - b. X
    - c. Y
    - d. Z

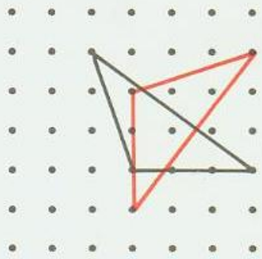


11. Trace $\overline{RS}$ and $\overline{VW}$. If $\overline{VW}$ is the image of $\overline{RS}$ under a rotation, locate the center of rotation, and find the angle of rotation.

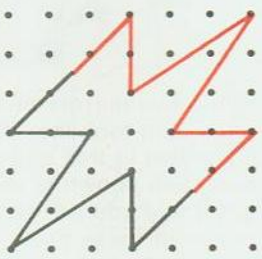


12. In each of the figures shown, the red figure is the image of the black figure under a turn. Label the center of the turn and indicate the angle measure for each turn. (Tracing paper may be helpful.)

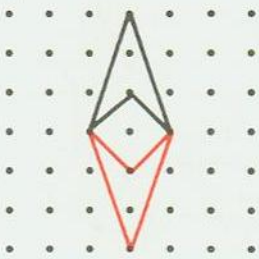
a.



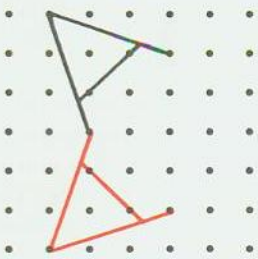
b.



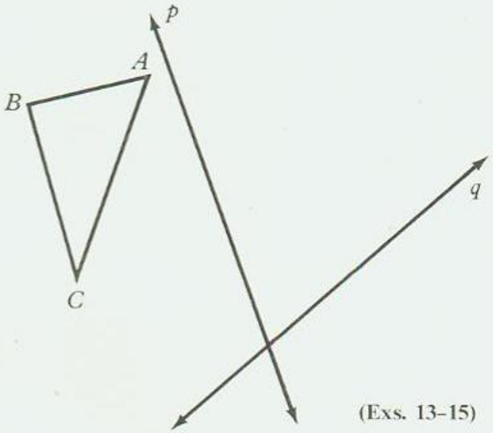
c.



d.



C.



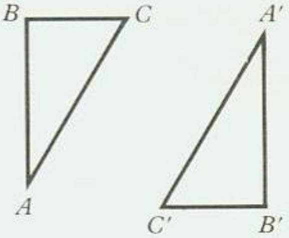
(Exs. 13–15)

13. Draw a pair of intersecting lines p and q and a triangle ABC as shown. Draw the image of $\triangle ABC$ for the reflection over line p followed by the reflection over line q. Call this image $\triangle A''B''C''$.

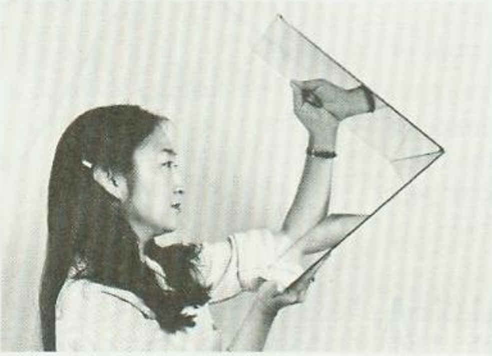
14. Use tracing paper to check that $\triangle A''B''C''$ that you drew in exercise 13 is the rotation image of $\triangle ABC$.

15. Measure with a protractor the acute angle formed by the lines p and q in your drawing from exercise 13. How does this angle compare in size with the angle of rotation?

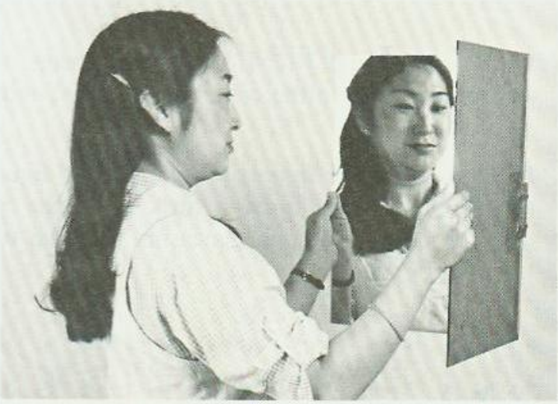
16. A 180° rotation about a point is called a *half-turn*. $\triangle A'B'C'$ is the half-turn rotation image of $\triangle ABC$. Trace these figures and find the center of the half-turn.



Activity



Hinge two mirrors at a 90° angle and look into them as shown. Are you surprised by what you see? Can you explain your observation?

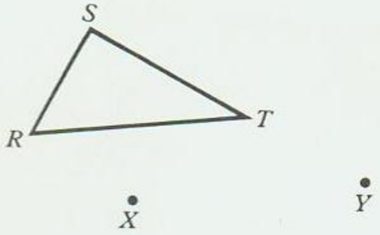


Hinge two mirrors at a 90° angle and look into them as shown in this photo. The reflection shown is one of three that you will see. How is one of these three images different from the other two?

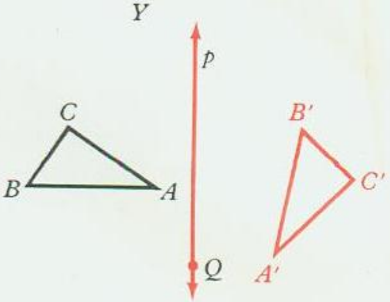
17. Mark points X and Y on your paper and draw $\triangle RST$.

a. Draw the half-turn image of $\triangle RST$ around X. Using the image of this turn, draw a half-turn around Y.

b. What single motion has the same effect as a half-turn about X followed by a half-turn about Y? Be as specific as possible in describing this motion.



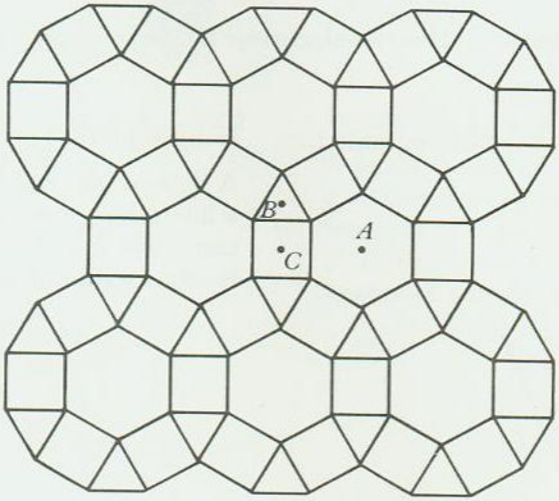
18. $\triangle A'B'C'$ is the image of $\triangle ABC$ under a rotation about center Q. Trace these figures. Can you construct line q so that a reflection over line p followed by a reflection over line q will have the same effect as the turn?



PROBLEM SOLVING

Trace the pattern below on tracing paper. Rotate the tracing 60° about point A and notice that the tracing matches the pattern (assuming that the pattern and its tracing are both extended in all directions to cover the entire plane).

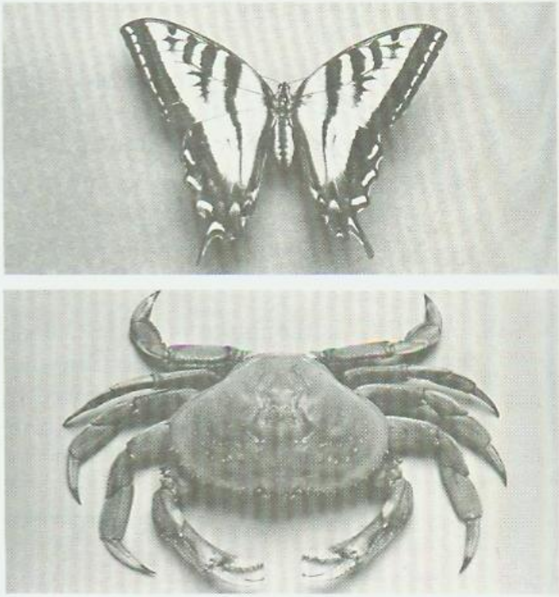
1. What other angles of rotation centered at A will make the pattern fit back on itself?
2. What are the angles of rotation centered at points B and C that will make the pattern fit back on itself?



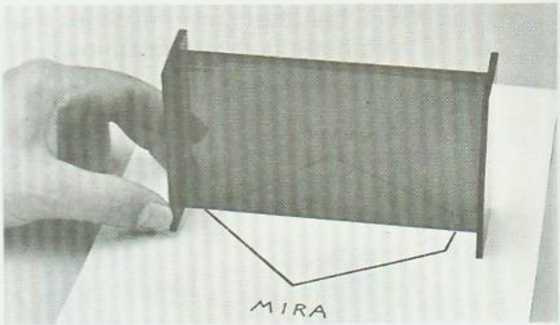
13–5 Symmetry

The butterfly and crab shown here possess a natural beauty related to their shape. One half appears identical to the other half. The match seems so perfect that a mirror could be placed so that one half is reflected giving the entire specimen.

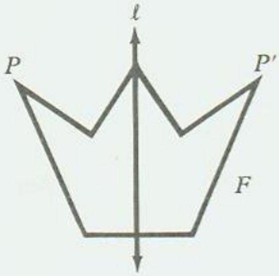
We say that the shapes possess *reflectional symmetry* and that the line on which the mirror is placed is a *line of symmetry*.



To test a figure for reflectional symmetry, you can trace it on a piece of paper and see if it can be folded so that one half exactly coincides with the other half. A "mirror" test can also be used, as shown in this picture. If a "mirror" can be placed on the figure so that half of the figure and its reflection appear to accurately form the entire figure, then the figure is said to possess reflectional symmetry.



We present the following definition, aided by this illustration.

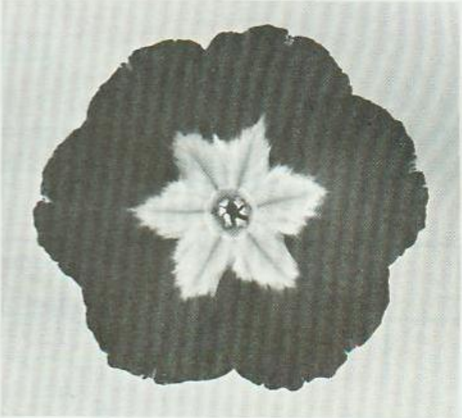


Definition 13-4

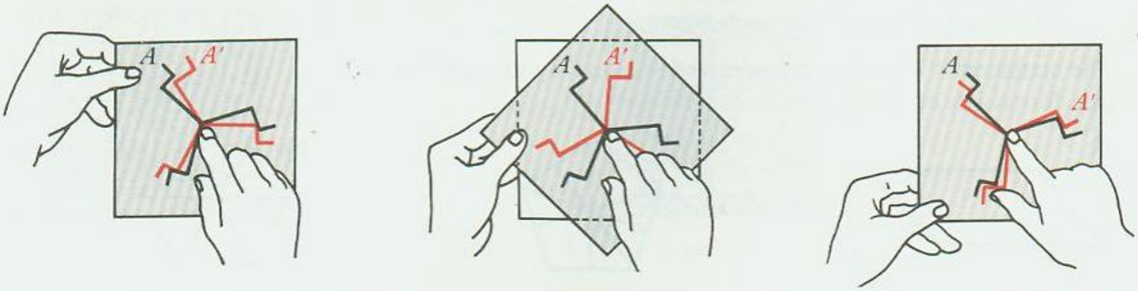
A figure F has **reflectional symmetry** if there is a line ℓ such that the reflection image over ℓ of each point P of F is also a point of F. The line ℓ is called a **line of symmetry** for F.

This flower possesses another kind of symmetry. It can be turned about a fixed center to new positions so as to appear almost identical to the way it appeared originally.

We say that the shape possesses *rotational symmetry* and that the fixed center is the *center of rotational symmetry.*

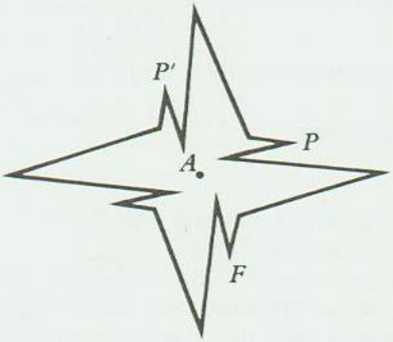


To test a figure for rotational symmetry, you can trace the figure on a piece of tracing paper or a piece of plastic. Place the tracing directly on top of the original figure. Then, while holding the center fixed, turn the tracing until the tracing and the original again coincide, as shown.



Checking for rotational symmetry—the "trace and turn test."

We present the following definition, aided by this illustration.



Definition 13–5

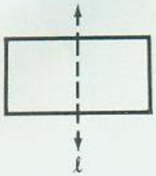
A figure F has **rotational symmetry** if there is a rotation about a center A such that the rotation image of each point P of figure F is also a point of F. The center, A, of the rotation is called the **center of rotational symmetry** for F.

EXERCISES

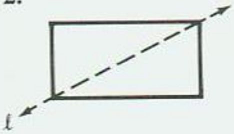
A.

In which of the following figures is ℓ a line of symmetry? Check your answers using the "mirror" test for reflectional symmetry.

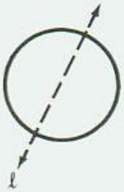
1.



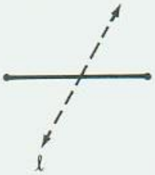
2.



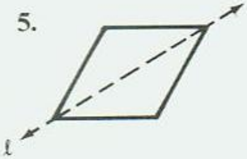
3.



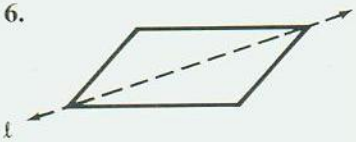
4.



5.



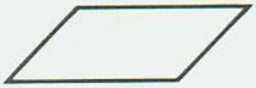
6.



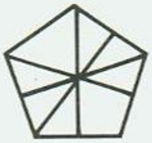
7. Draw a square, a rhombus, a rectangle, a trapezoid, a regular pentagon, and a regular hexagon and show all lines of reflectional symmetry for each figure.

The statement "This figure possesses rotational symmetry." is true for which of the figures in exercises 8–10?

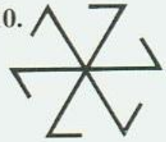
8.



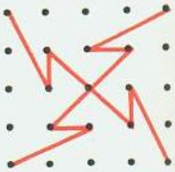
9.



10.

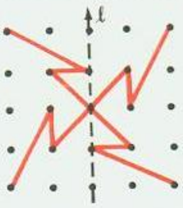


Activity



Pictured is a figure with 90° rotational symmetry (can be rotated 90° to fold back on itself) on a 5 × 5 geoboard. The figure is constructed by drawing four identical paths that satisfy these conditions.

1. All paths begin at the center nail and move from nail to nail until reaching an outside boundary nail.
2. None of the four paths touch or cross each other or themselves.
3. Once a path reaches an outside boundary it stops.



Draw on 5 × 5 dot paper at least 20 different patterns with 90° rotational symmetry that satisfy the conditions above.
Note that the figure at the left is not considered to be "different" from the one above, since it is the figure above reflected over line ℓ. Figures that can be reflected or rotated to match each other are not considered different.

B.

Draw figures satisfying each of the following conditions.

11. Nonconvex quadrilateral with a line of symmetry

12. A hexagon with exactly two lines of symmetry

13. A hexagon with exactly three lines of symmetry

14. A pentagon with exactly one line of symmetry

15. A figure with an infinite number of lines of symmetry

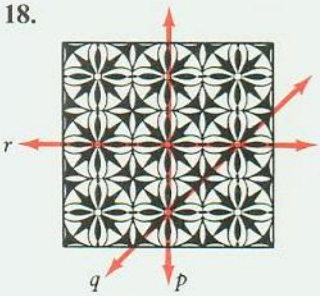
16. Sketch a polygon that possesses rotational symmetry but does not possess reflectional symmetry.

17. Sketch a polygon which possesses both rotational and reflectional symmetry.

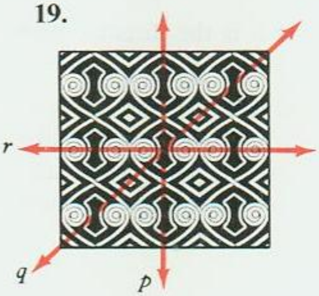
C.

For exercises 18 and 19, assume that the pattern has been extended to cover the entire plane. In each problem which of the lines p, q, and r are lines of symmetry?

18.



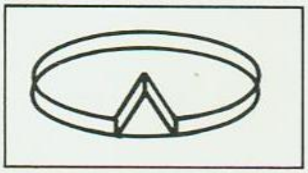
19.



PROBLEM SOLVING

NOW NO
SWIMS
ON MON

A sign like this appeared beside a public swimming pool. Turn this card 180°. What do you notice?



A slice is missing from this cake. Can you turn the card in such a way as to find the slice?

337-31770

Oliver Lee asked that the above number appear on his license plate. Turn the card 180°. Can you explain why he made this request?

Important Ideas—Chapter 13

Terms

Reflection (p. 476)
Translation image (p. 484)
Rotation (p. 488)
Reflectional symmetry (p. 494)
Line of symmetry (p. 494)
Rotational symmetry (p. 495)
Center of rotational symmetry (p. 495)

Theorems

13-1 Given a line reflection:
  a. the reflection image of a segment is a segment of equal length;
  b. the reflection image of an angle is the angle of equal measure.

13-2 If lines r and s are parallel, then a reflection over line r followed by a reflection over line s is a translation. Furthermore, if A'' is the image of A, then
  a. $\overline{AA''} \perp r$,
  b. $AA'' = 2d$ where d is the distance between the lines r and s.

13-3 If lines r and s intersect in point O, then a reflection over t followed by a reflection over s is a rotation. Point O is the center of rotation and the angle of rotation is 2α, where α is the measure of the acute angle between lines r and s.

Chapter 13—Review

1. Indicate on your paper whether the following statements are true or false.

   a. Every reflection is a transformation.

   b. If P' is the image of P for a given reflection over line ℓ, then ℓ is the perpendicular bisector of PP'.

   c. A rotation can be represented as two reflections.

   d. A square has exactly two lines of symmetry.

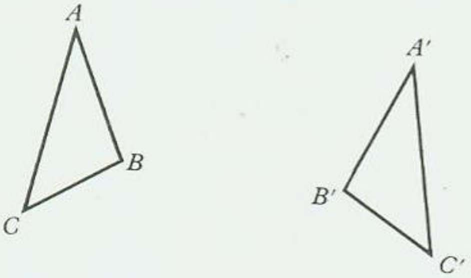
   e. An isosceles trapezoid has rotational symmetry.

2. Name two capital letters that have exactly two lines of symmetry? Are there any others?

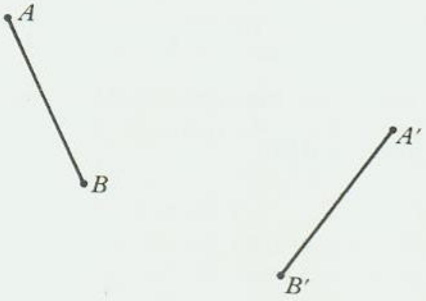
3. Name two capital letters that have rotational symmetry but no line symmetry. Are there any others?

4. How many lines of symmetry does a regular octagon have?

5. Trace the triangles ABC and $A'B'C'$. Construct line ℓ so that $\triangle A'B'C'$ is the reflection image of $\triangle ABC$ over ℓ.

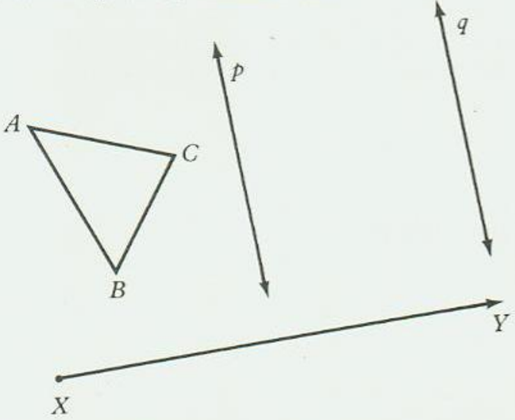


6. Trace $\overline{AB}$ and $\overline{A'B'}$. Find the center of rotation and the angle of rotation if $\overline{A'B'}$ is the image of $\overline{AB}$ under a rotation.

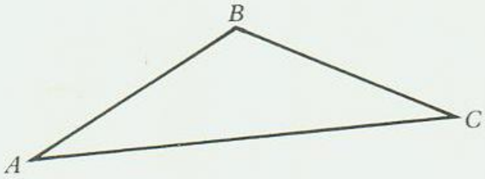


Chapter 13—Test

1. Indicate on your paper whether the following statements are true or false.
   - **a.** Every translation is a transformation.
   - **b.** If $\overline{A'B'}$ is the image of $\overline{AB}$ under a rotation about center O, then $m\angle AOA' = m\angle BOB'$.
   - **c.** A translation can be represented as two reflections.
   - **d.** A parallelogram has exactly two lines of symmetry.
   - **e.** A rhombus has rotational symmetry.
2. Name two capital letters that have exactly one line of symmetry. Are there any others?
3. Name two capital letters that have both rotational and reflectional symmetry. Are there any others?
4. Does a regular polygon have to have both rotational and reflection symmetry? Explain.
5. Trace the triangle ABC, lines p and q, and the translation arrow $\overrightarrow{XY}$. Reflect $\triangle ABC$ over p and over q. How does the distance between p and q relate to XY?



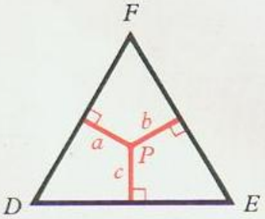
6. Trace $\triangle ABC$. Using C as the center of rotation, rotate $\triangle ABC$ through an angle of 120°.



Problem Solving Techniques

Examine Special Cases

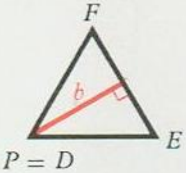
Sometimes it is helpful to consider special cases when solving problems. Study the example below.

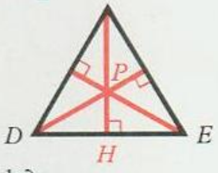


Example

Let P be any point on or inside an equilateral triangle. If a, b, and c are the lengths of the perpendicular segments from P to the sides of the triangle, how does the sum $a + b + c$ compare with the length of an altitude?

Consider these special cases.



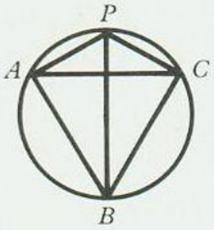


- **i.** Suppose $P = D$. Then $a = c = 0$. Since b is then the altitude to $\overline{FE}$, $a + b + c$ equals the length of the altitude.
- **ii.** Suppose P is at the center of $\triangle DEF$. Then $a = b = c = \frac{1}{3}FH$ (the altitude). Then $a + b + c = FH$ or the length of the altitude.

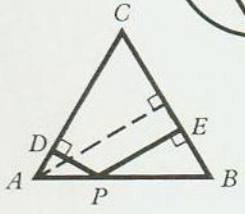
Does it appear that $a + b + c$ is equal to the length of the altitude?

PROBLEMS

1. In the example above try the special case when P is the midpoint of a side. Does $a + b + c$ equal the length of the altitude for this case also?



2. $\triangle ABC$ is an equilateral triangle inscribed in a circle. P is any point on the circle. How are PA, PB, and PC related? (*Hint:* Consider these cases: **i.** $P = A$, **ii.** $P = C$, and **iii.** $PA = PC$.)



3. **Given:** $AB = BC = AC = 4$ cm. P is any point on $\overline{AB}$.
   $\overline{DP} \perp \overline{AC}$, $\overline{PE} \perp \overline{BC}$.
   Find: $DP + PE$
   (*Hint:* Consider the relationship among DP, PE, and AF using the special cases when $P = A$ and $PA = PB$.)

4. **Given:** Square $ABCD$ with each side 1 unit in length.
   Find: A point P such that $PA + PB + PC + PD$ is a minimum.

   (*Hint:* Consider these cases: **i.** $P = A$, **ii.** P is the midpoint of $\overline{AB}$, **iii.** P is intersection of the diagonals, and **iv.** Use a ruler to find $PA + PB + PC + PD$ when P is in the interior of $ABCD$ but not at the point of intersection of the diagonals.)

   Which position seems to produce a minimum for $PA + PB + PC + PD$?